MATH 5061 Problem Set 2[1]

Due date: Feb 26, 2024

Problems: (Please hand in your assignments by submitting your PDF via email. **Late submissions will not be accepted.**)

Throughout this assignment, we use (M, g) to denote a smooth n-dimensional Riemannian manifold with its Levi-Civita connection ∇ unless otherwise stated. The Riemann curvature tensor (as a $(0,4)$-tensor) of (M, g) is denoted by R.

1. Prove that the antipodal map $A(p) = -p$ induces an isometry on $\mathbb{S}^n$. Use this to introduce a Riemannian metric on $\mathbb{RP}^n$ such that the projection map $\pi : \mathbb{S}^n \to \mathbb{RP}^n$ is a local isometry.

2. Show that the isometry group of $\mathbb{S}^n$, with the induced metric from $\mathbb{R}^{n+1}$, is the orthogonal group $O(n+1)$.

3. For any smooth curve $c : I \to M$ and $t_0, t \in I$, we denote the parallel transport map as $P = P_{c,t_0,t} : T_{c(t_0)}M \to T_{c(t)}M$ along c from $c(t_0)$ to $c(t)$.

   (a) Show that P is a linear isometry. Moreover, if M is oriented, then P is also orientation-preserving.

   (b) Let X, Y be vector fields on M, $p \in M$. Suppose $c : I \to M$ is an integral curve of X with $c(t_0) = p$. Prove that
   $$(\nabla_X Y)(p) = \frac{d}{dt}\bigg|_{t=t_0} P^{-1}_{c,t_0,t}(Y(c(t))).$$

4. Prove the *second Bianchi identity*: for any vector fields $X, Y, Z, W, T \in \Gamma(TM)$,
$$(\nabla_X R)(Y, Z, W, T) + (\nabla_Y R)(Z, X, W, T) + (\nabla_Z R)(X, Y, W, T) = 0.$$

[1]Last revised on February 3, 2024